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Ascending auctions with bidder asymmetries

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We present a partial identification approach for ascending auctions with bidder asymmetries, where bidders' asymmetric types may be *unobservable* to the econometrician. Our approach yields sharp bounds and builds on and generalizes other recent bounds approaches for correlated private values ascending auctions. When bidder identities are *observable*, our approach yields tighter bounds than previous approaches that ignore asymmetry, demonstrating that bidder asymmetries can function as an aid rather than a hindrance to identification. We present a nonparametric estimation and inference approach relying on our identification argument and apply it to data from U.S. timber auctions, finding that bounds on optimal reserve prices and other objects of interest are noticeably tighter when exploiting bidder asymmetries.

Keywords. Ascending auction, partial identification, correlated values, asymmetries.

JEL classification. C10, D44, L10.

1. Introduction

Identification of bidder valuations in ascending auctions faces a variety of challenges: the dropout point of the highest-value bidder is never observed, bidding may not follow a button auction model, valuations may be correlated, and bidders may be asymmetric—potentially with their asymmetric types or identities being unobservable to the econometrician. Consequently, little empirical work on ascending auctions has

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been done outside of the symmetric independent private values (IPV), button auction framework. Recent advances in the literature have developed bounds approaches that relax the button auction assumption (Haile and Tamer (2003)) or the independence assumption (Aradillas-López, Gandhi, and Quint (2013)) but maintain the assumption of symmetry; other novel advances have relaxed the symmetry or independence assumptions but require that information about bidders' asymmetric types be observable by the econometrician (Komarova (2013a)).

This paper takes a uniquely different approach from the previous literature, demonstrating that in private values ascending auctions, even when allowing for values to be correlated, relaxing the assumption of bidder symmetry need not complicate identification or estimation, and can in fact *aid* identification and estimation. This arises because, unlike first price auctions, where equilibrium bidding strategies are affected by bidder asymmetries, bidding in private values ascending auctions need not be. The approach in this paper is also unique in that we consider cases where bidders' asymmetric types may be *unobservable* to the econometrician. We demonstrate that for certain identification arguments in ascending auctions unobservable asymmetries can be ignored. We demonstrate further that when bidder types are *observed* exploiting these asymmetries yields tighter bounds on objects of interest, such as buyer and seller surplus or the optimal reserve price. Finally, we derive a precise sufficient condition, new to the literature, on the composition of types participating in the auction such that the researcher may exploit exogenous variation in the *number* of bidders even in asymmetric settings. This condition has easily testable implications which we verify in our empirical application.

In this paper, we focus on the setting of Aradillas-López, Gandhi, and Quint (2013) (AGQ), although the ideas behind our approach could also apply in other settings, such as Haile and Tamer (2003). AGQ modeled a symmetric private values setting and demonstrated that buyer and seller surplus depend only on the marginal distributions of the highest and second-highest order statistics, and that, even under correlation, order statistics relationships can be used to obtain bounds on buyer and seller surplus when only the transaction price and the number of bidders is observed. We demonstrate that their approach applies to more general settings than symmetric correlated private values. In particular, we show that all their bounds on buyer surplus, seller surplus, and the optimal reserve price hold without modification if bidders are asymmetric with correlated private values, even if bidder identities are *unobserved*. When bidder identities are *observed*, we derive new bounds, which will typically be tighter than AGQ's. All of our results apply to the most general type of bidder asymmetries, where the joint distribution of buyer valuations is unrestricted, and the bounds we derive are sharp.

In addition to our nonparametric identification arguments, we present an estimation approach that allows the researcher to control for auction-level heterogeneity fully nonparametrically. We illustrate the approach using U.S. timber auction data, where the common categorization of bidder asymmetries is between mills and loggers (e.g., Athey, Levin, and Seira (2011), Athey, Coey, and Levin (2013), Roberts and Sweeting (2016)). We demonstrate that, relative to previous approaches that treat bidders as symmetric, exploiting this dimension of bidder asymmetry leads to tighter bounds on objects of interest, including the estimated distribution of the willingness-to-pay of the highest-value bidder, seller surplus, and the optimal reserve price.

To our knowledge, these are some of the first positive identification results for ascending auctions with asymmetric correlated values that do not rely on the existence of bidder-specific covariates for all bidders.[1] Such covariates may not be readily available in practice, and our bounds still apply in these settings. Komarova (2013a) is one exception to this. She provides several identification arguments for asymmetric correlated private values ascending auctions, requiring that bidder identities or types be observable. Athey and Haile (2002) and Komarova (2013b) provide identification arguments for asymmetric IPV ascending auctions. Our results are the first identification results of which we are aware for ascending auctions with bidder asymmetries and *unobserved* bidder identities or types. Lamy (2012) presents results on unobserved bidder identities in asymmetric first price auctions.

The remainder of the paper is as follows. In Section 2, we introduce the AGQ framework. In Section 3, we present a general asymmetric version of this framework and demonstrate how bounds on order statistic distributions and, consequently, on bidder and seller surplus and the optimal reserve price, can be obtained when bidder asymmetries are unobservable and how these bounds can be improved when bidder asymmetries are observable. We describe the nonparametric estimation approach in Section 4. Section 5 presents the empirical application and Section 6 concludes. Replication file are available in a supplementary file on the journal website, http://qeconomics.org/supp/474/code_and_data.zip.

2. Baseline model: Symmetric bidders

We first present the AGQ framework and then show how it is nested by our more general model that allows for unobservable or observable bidder asymmetries. The identification results of AGQ are stated under the following assumptions and definitions.

Assumption 1. *Bidders have symmetric private values.*

Assumption 2. *The transaction price in an auction is the greater of the reserve price and the second-highest bidder's willingness to pay.*

Assumption 3. *Let N be a random variable denoting the number of bidders in an auction, with n representing realizations of N. For each n in the support of N, the joint distribution $\mathbf{F}^n$ of private values $(V_1, V_2, \dots, V_n)$ is such that for any v and i, the probability $\Pr(V_i < v | N = n, |\{j \neq i : V_j < v\}| = k)$ is nondecreasing in k.*

Definition 1. Let $\mathbf{F}_m^n$ be the joint distribution of m randomly chosen bidders in an n-bidder auction. *Valuations are independent of N* if $\mathbf{F}_m^n = \mathbf{F}_m^{n'}$ for any $m \leq n, n'$.

[1] Athey and Haile (2007, Theorem 6.1) describe an identification-at-infinity argument showing how variation in bidder-specific covariates can be used to recover the distribution of bidder values. Somaini (2011) demonstrates that bidder asymmetries, along with observable, bidder-specific covariates, can also be advantageous in first price auction settings with common values.

Assumption 1 will be relaxed in the following section to allow for asymmetric bidders. Assumption 2 is the button auction assumption, which is relaxed by Haile and Tamer (2003) in their IPV framework (AGQ also contains a discussion of how this assumption may be relaxed in their framework). Assumption 3 nests several well known information settings, such as affiliated private values, conditional IPV, and auctions with unobserved, auction-level heterogeneity.

Under these assumptions, AGQ derive two core results. First, Assumptions 1 and 3 imply that, for any n and v,

$$F_{n:n}(v) \geq \phi_n\big(F_{n-1:n}(v)\big)^n, \tag{1}$$

where $\phi_n : [0, 1] \to [0, 1]$ is the inverse of the mapping from p to $np^{n-1} - (n-1)p^n$, and where $F_{m:n}$ denotes the marginal distribution of the mth order statistic. Second, when valuations are independent of N, then, for a fixed n and $\bar{n} > n$ and for any v,

$$F_{n:n}(v) = \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}:\bar{n}}(v). \tag{2}$$

AGQ demonstrate further that the equality (2) becomes a "greater than or equal to" ($\geq$) when valuations are stochastically increasing in N, defined as follows.

Definition 2. Let $F^n_{m:m}$ denote the marginal distribution of the maximum order statistic among m bidders chosen at random from n bidder auctions. Valuations are *stochastically increasing in N* if $n > n'$ implies that $F^n_{m:m}$ first order stochastically dominates $F^{n'}_{m:m}$ for any $m \leq n'$.

AGQ use the lower bound implied by (1), the upper bound implied by the fact that, for any n and v, $F_{n:n}(v) \leq F_{n-1:n}(v)$, and the order statistics relationship in (2) to bound $F_{n:n}$ as

$$F_{n:n}(v) \leq \bar{F}_{n:n}(v) \equiv \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}-1:\bar{n}}(v), \tag{3}$$

$$F_{n:n}(v) \geq \underline{F}_{n:n}(v) \equiv \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} \phi_{\bar{n}}\big(F_{\bar{n}-1:\bar{n}}(v)\big)^{\bar{n}}. \tag{4}$$

These bounds on $F_{n:n}(v)$ are then used to derive bounds on buyer and seller surplus,

$$\pi_n(r) \geq \underline{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v) - v_0 - \bar{F}_{n:n}(r) \cdot (r - v_0), \tag{5}$$

$$\pi_n(r) \leq \bar{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v) - v_0 - \underline{F}_{n:n}(r) \cdot (r - v_0), \tag{6}$$

$$BS_n(r) \geq \underline{BS}_n(r) \equiv \int_0^\infty \max\{r, v\}\, d\bar{F}_{n:n}(v) - \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v), \tag{7}$$

$$BS_n(r) \leq \overline{BS}_n(r) \equiv \int_0^\infty \max\{r, v\}\, d\underline{F}_{n:n}(v) - \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v), \tag{8}$$

where v_0 represents the value of the good to the seller (assumed to be common knowledge). Furthermore, AGQ demonstrate that $\max_r \pi_n(r) \in [\max_r \underline{\pi}_n(r), \max_r \bar{\pi}_n(r)]$ and $\arg\max_r \pi_n(r) \in \{r : \bar{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\}$.

Importantly, these bounds only depend on the distribution of the second order statistic, which, by Assumption 2, the econometrician observes when reserve prices are nonbinding. When valuations are independent of N, the bounds are two sided, and when valuations are stochastically increasing in N, only the upper bounds hold. These bounds are sharp: the upper bounds on buyer and seller surplus correspond to the case of independence of bidders' valuations and the lower bound corresponds to the case of perfect correlation of bidders' valuations.

We follow AGQ in maintaining Assumption 2 throughout. We demonstrate how AGQ's bounds generalize when Assumption 1 is relaxed to allow for asymmetric bidders, even when the types of the bidders are unobserved, and we present bounds that improve on the AGQ bounds when bidder types are observed.

3. Asymmetric bidders

3.1 *Definition of symmetry*

Let $\mathbb{N}$ be the full set of potential bidders. Let $\mathcal{P}$ be a random vector representing the identities or types of bidders participating in an auction, with realizations $P \subset \mathbb{N}$. As in Section 2, let N be a random variable representing the number of bidders participating in an auction, with realizations $n \in \mathbb{N}$. When necessary to clarify the number of bidders in a set of participating bidders, we let P_n denote an arbitrary set of n participating bidders. Define $\mathbf{F}^P$ to be the joint distribution of $(V_i)_{i\in P}$ when P is the set of participating bidders.[2] As in Section 2, $\mathbf{F}^n$ represents the joint distribution of values conditional on there being n entrants, but unconditional on the set of participants. Therefore, $\mathbf{F}^n(v_1 \cdots v_n) = \sum_{P_n \subset \mathbb{N}} \Pr(\mathcal{P} = P_n | N = n) \mathbf{F}^{P_n}(v_1 \cdots v_n)$.

Following AGQ, we use the term *bidder symmetry* to mean *exchangeability*, defined below.

Definition 3. Bidders are *exchangeable* if $\mathbf{F}^n(v_1, \dots, v_n) = \mathbf{F}^n(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ for any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ and any $(v_1, \dots, v_n)$.

We will say that a subset of bidders are of the same *type* if they are exchangeable. All the results we derive below apply in the most general case of bidder asymmetries, where each unique bidder is potentially a unique, asymmetric type. The results also apply in cases where multiple bidders are of the same type, such as in the timber auctions example in Section 5, where we categorize bidders as mills or loggers. For example, in a setting in which each bidder identity is a distinct bidder type and $\mathbb{N} = \{1, 2, \dots, 10\}$, one possible five-bidder set of participating bidders is $P_5 = \{1, 2, 5, 8, 9\}$. If $\mathbb{N}$ again contains ten

[2] We adopt the convention that bidders are ordered according to their identities, that is, if $P = \{2, 5, 12\}$, then $\mathbf{F}^P$ is the joint distribution of (V_2, V_5, V_{12}), rather than, for example, the joint distribution of (V_5, V_2, V_{12}).

bidders but these bidders are only of two types, high (H) and low (L), with five bidders of each type, then one possible five-bidder set of participating bidders is $P_5 = \{3H, 2L\}$.

To show that the AGQ approach applies without symmetry (i.e., exchangeability), we first prove that an analog of (1) holds, and then give natural sufficient conditions for (2). Finally, we combine these results to obtain the required bounds.

3.2 *Bounds on the maximum order statistic distribution with bidder asymmetries*

We show that an analog of (1) holds, even without bidder exchangeability. The intuition behind this result is that one can randomly permute bidders' values, which preserves order statistics and yields an exchangeable random vector, allowing us to then leverage one of AGQ's technical lemmas. Our relaxation of bidder symmetry requires a slight modification of Assumption 3.

Assumption 3′. *For each set $P_n \subset \mathbb{N}$, $\mathbf{F}^{P_n}$ is such that for any v and i, $C \subset C' \subseteq P_n$ implies* $\Pr(V_i < v|\mathcal{P} = P_n, \{j \neq i : V_j < v\} = C) \leq \Pr(V_i < v|\mathcal{P} = P_n, \{j \neq i : V_j < v\} = C')$.

Let $F^{P_n}_{m:n}$ be the marginal distribution of the mth order statistic of valuations when the set of participating bidders is P_n. As in Section 2, $F_{m:n}$ represents the distribution of the mth order statistic of values conditional on there being n entrants, but unconditional on the set of participants. Therefore, $F_{m:n}(v) = \sum_{P_n \subset \mathbb{N}} \Pr(\mathcal{P} = P_n|N = n)F^{P_n}_{m:n}(v)$.

Lemma 1. *If Assumption 3′ holds, then for any v and n, $F_{n:n}(v) \in [E_{\mathcal{P}}(\phi_n(F^{\mathcal{P}}_{n-1:n}(v))^n|N = n), F_{n-1:n}(v)]$.*

Proof. We first prove that, for any P_n, $F^{P_n}_{n:n}(v) \in [\phi_n(F^{P_n}_{n-1:n}(v))^n, F^{P_n}_{n-1:n}(v)]$; the desired result follows by taking expectations with respect to $\mathcal{P}$ conditional on $N = n$. To prove this intermediate result, we create an exchangeable random vector whose order statistics have the same distribution as the order statistics of $(V_i)_{i \in P_n}$, and we show that under Assumption 3′ it satisfies AGQ's Assumption 3.

Suppose without loss of generality that $P_n = \{1, \dots, n\}$. We condition on the event $\mathcal{P} = P_n$ throughout the remainder of this proof and omit it from the notation. Denote the $n!$ possible permutation functions of n elements by $\sigma_1, \dots, \sigma_{n!}$. Let $\mathbf{H}^{P_n}(v_1, \dots, v_n) = \frac{1}{n!}\sum_{t=1}^{n!} \mathbf{F}^{P_n}(v_{\sigma_t(1)}, \dots, v_{\sigma_t(n)})$. Let $(U_1, \dots, U_n)$ be a random vector with cumulative distribution function $\mathbf{H}^{P_n}$, and note that this random vector is exchangeable. We now prove that it satisfies Assumption 3. For any v, i, and C, we have

$$\Pr\bigl(U_i < v|\{j \neq i : U_j < v\} = C\bigr) = \frac{1}{n!}\sum_{t=1}^{n!} \Pr\bigl(V_{\sigma_t(i)} < v|\{j \neq i : V_{\sigma_t(j)} < v\} = C\bigr).$$

By Assumption 3′, the summand above is nondecreasing in C for each σ_t, so the sum is nondecreasing in C. That is, $C \subset C'$ implies $\Pr(U_i < v|\{j \neq i : U_j < v\} = C) \leq \Pr(U_i < v|\{j \neq i : U_j < v\} = C')$. Consequently, $\Pr(U_i < v||\{j \neq i : U_j < v\}| = k)$ is nondecreasing in k. Thus the random vector $(U_1, \dots, U_n)$ satisfies Assumption 3. Denote the distribution of its mth order statistic by $H^{P_n}_{m:n}$. Applying AGQ's Lemma 3,

$H_{n:n}^{P_n}(v) \in [\phi_n(H_{n-1:n}^{P_n})^n, H_{n-1:n}^{P_n}(v)]$. The distributions of order statistics of $(V_1, \dots, V_n)$ and $(U_1, \dots, U_n)$ are identical, so $F_{m:n}^{P_n}(v) = H_{m:n}^{P_n}(v)$ for $m \le n$ and the result follows. $\square$

The lower bound proposed in Lemma 1, $E_{\mathcal{P}}(\phi_n(F_{n-1:n}^{\mathcal{P}}(v))^n | N = n)$, is an expectation over all possible sets of participants P of n bidders, while the upper bound is the same as in AGQ. Our next lemma shows that this new lower bound on $F_{n:n}(v)$ (used below to obtain upper bounds on buyer and seller surplus) is no smaller than AGQ's, and is strictly larger when $F_{n-1:n}^{\mathcal{P}}(v)$ is not almost surely constant with respect to $\mathcal{P}$.

Lemma 2. *For all $n \ge 2$ and all v, $\phi_n(F_{n-1:n}(v))^n \le E_{\mathcal{P}}(\phi_n(F_{n-1:n}^{\mathcal{P}}(v))^n | N = n)$. If $F_{n-1:n}^{\mathcal{P}}(v)$ is not almost surely constant with respect to $\mathcal{P}$, then $\phi_n(F_{n-1:n}(v))^n < E_{\mathcal{P}}(\phi_n(F_{n-1:n}^{\mathcal{P}}(v))^n | N = n)$.*

Proof. We apply Jensen's inequality. Define the function $g : [0, 1] \to [0, 1]$ by $g(x) = \phi_n(x)^n$. Recall that $\phi_n^{-1}(p) = np^{n-1} - (n-1)p^n$. By construction, $g^{-1}(p) = \phi_n^{-1}(p^{1/n})$ so that $g^{-1}(p) = np^{(n-1)/n} - (n-1)p$. The function g^{-1} is increasing and strictly concave on $[0, 1]$, so g is strictly convex on $[0, 1]$. By Jensen's inequality, $g(E_{\mathcal{P}}(F_{n-1:n}^{\mathcal{P}}(v) | N = n)) \le E_{\mathcal{P}}(g(F_{n-1:n}^{\mathcal{P}}(v)) | N = n)$, with strict inequality unless $F_{n-1:n}^{\mathcal{P}}(v)$ is almost surely constant with respect to $\mathcal{P}$. $\square$

Lemmas 1 and 2 establish a lower bound on $F_{n:n}$ that is generally tighter than the AGQ lower bound when bidder identities or types are observable. Moreover, when bidder identities or types are not observable to the researcher (and hence $E_{\mathcal{P}}(\phi_n(F_{n-1:n}^{\mathcal{P}}(v))^n | N = n)$ cannot be computed) they imply that the AGQ bounds, $[\phi_n(F_{n-1:n}(v))^n, F_{n-1:n}(v)]$, are still valid bounds even with nonexchangeable bidders.

For a simple numerical illustration of Lemma 2, consider a setting with two bidder types. Type H has value 1 and type L has value 0. With equal probability, either two type H bidders enter or two type L bidders enter, so that $\Pr(\mathcal{P} = \{H, H\}) = \Pr(\mathcal{P} = \{L, L\}) = 0.5$. For any $v \in (0, 1)$, $E_{\mathcal{P}}(\phi_2(F_{1:2}^{\mathcal{P}}(v))^2 | N = 2) = 0.5 \cdot \phi_2(0)^2 + 0.5 \cdot \phi_2(1)^2 = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$. By contrast, $\phi_2(F_{1:2}(v))^2 = \phi_2(0.5)^2 \approx 0.086$. Thus, taking into account asymmetries among bidders can lead to a much higher lower bound (0.5) than the value given by ignoring these asymmetries (0.086).

3.3 *Sufficient conditions for valuations independent of N with bidder asymmetries*

Having shown that (1) holds without exchangeability, and that, in fact, a tighter bound than (1) holds when bidder types are observable, we now provide sufficient conditions for (2) to hold. In doing so, we introduce some additional notation. For $P' \subset P$, let $\mathbf{F}^{P'|P}$ denote the joint distribution of $(V_i)_{i \in P'}$ in auctions where P is the set of participants. For $r \le m < n$ and $P_m \subset P_n$, let $F_{r:m}^{P_m|P_n}$ be the marginal distribution of the $(r:m)$ order statistic of $(V_i)_{i \in P_m}$ in auctions where P_n are the participants. Let $F_{r:m}^{P_n}$ denote the marginal distribution of the $(r:m)$ order statistic in auctions where P_n are the participants and all but m bidders have been removed at random. Therefore,

$$F_{r:m}^{P_n}(v) \equiv \frac{1}{\binom{n}{m}} \sum_{P_m \subset P_n} F_{r:m}^{P_m|P_n}(v). \tag{9}$$

Define $F^n_{r:m}$ to be distribution of the $(r:m)$ order statistic of m randomly chosen bidders from all n bidder auctions unconditional on the set of participants, so that $F^n_{r:m}(v) = \sum_{P_n \subset \mathbb{N}} \Pr(\mathcal{P} = P_n|N = n) F^{P_n}_{r:m}(v)$. Let $\Pr(P_n|P_{n+1})$ denote the probability that P_n would be obtained by dropping a bidder at random from P_{n+1}.[3]

For (2) to hold, it suffices that $F_{n:n} = F^{n+1}_{n:n}$. AGQ assume this is the case; that is, they assume that valuations are independent of N (Definition 1). While this is a natural assumption when bidders are symmetric, it is less clear what it entails in the asymmetric case. In particular, it is not obvious how it restricts participation of different types of bidders and their valuations conditional on participation or how it relates to existing definitions of exogenous participation with asymmetric bidders (e.g., Athey and Haile (2002)). Instead of assuming $F_{n:n} = F^{n+1}_{n:n}$, we give more primitive conditions on bidder participation and valuations that imply it. To this end, we modify Definitions 1 and 2 for the asymmetric case, and introduce a new definition.

Definition 1′. *Valuations are independent of supersets* if for all $P' \subset P$, $\mathbf{F}^{P'|P} = \mathbf{F}^{P'}$.

Definition 2′. *Valuations are stochastically increasing in supersets* if $P_n \supset P_{n'}$ implies that $F^{P_n}_{m:m}$ first order stochastically dominates $F^{P_{n'}}_{m:m}$ for any $m \le n'$.

Definition 4. *Bidder types are independent of* N if, for all P_n, $\Pr(\mathcal{P} = P_n|N = n) = \sum_{P_{n+1} \supset P_n} \Pr(P_n|P_{n+1}) \Pr(\mathcal{P} = P_{n+1}|N = n+1)$.

Definitions 1′ and 4 describe different kinds of exogeneity. Definition 1′ requires that conditional on some set of bidders participating, those bidders' values are independent of which other bidders participate (what Athey and Haile (2002) refer to as exogenous participation). Definition 4 is new to the literature, requiring that the distribution of participating bidder types in n bidder auctions is just like the distribution of participating bidder types in $n+1$ bidder auctions, with one bidder randomly removed. It restricts who participates, but not what their values are. Also, it is important to note that Definition 4 is much weaker than equal mixing over bidder types; the condition allows for the distribution of bidder types to be unrestricted within a given auction size n, and only requires that this distribution be the same *across n*.

To further clarify the meaning of Definition 4, we observe that one direct implication of bidder types being independent of N is that for any bidder type τ the expected fraction of bidders who are of type τ should be constant across N. We state this result as a lemma.

Lemma 3. *If bidder types are independent of* N, *then for any bidder type* τ *the expected fraction of bidders who are of type* τ *in auctions with* n *bidders is the same as in auctions with* n' *bidders for all* n, n'.

[3]For example, consider a case with two types, H and L. Then $\Pr(\{2H, 2L\}|\{3H, 2L\}) = \frac{3}{5}$, $\Pr(\{3H, 2L\}|\{3H, 3L\}) = \frac{1}{2}$, and so forth. If instead each bidder is a distinct type, $\Pr(P_n|P_{n+1}) = \frac{1}{n+1}$ for all n. To see this, fix P_n and note that for each $P_{n+1} \supset P_n$, P_n is obtained by dropping the bidder $P_{n+1} \setminus P_n$ from P_{n+1}. When bidders are dropped uniformly at random, this occurs with probability $\frac{1}{n+1}$.

Proof. Let τ be a fixed bidder type and let $\Pr(i=\tau|i\in P_n)$ be the probability that a bidder i randomly selected from a set P_n is of type τ. By taking a weighted sum of the left-hand and right-hand sides of the condition for bidder types being independent of N, one obtains

$$\begin{aligned}
&\sum_{P_n}\Pr(i=\tau|i\in P_n)\Pr(\mathcal{P}=P_n|N=n)\\
&\quad=\sum_{P_n}\Pr(i=\tau|i\in P_n)\sum_{P_{n+1}\supset P_n}\Pr(P_n|P_{n+1})\Pr(\mathcal{P}=P_{n+1}|N=n+1)\\
&\quad=\sum_{P_{n+1}}\Pr(\mathcal{P}=P_{n+1}|N=n+1)\sum_{P_n\subset P_{n+1}}\Pr(i=\tau|i\in P_n)\Pr(P_n|P_{n+1}).
\end{aligned}\tag{10}$$

The left-hand side of this weighted sum, $\sum_{P_n}\Pr(i=\tau|i\in P_n)\Pr(\mathcal{P}=P_n|N=n)$, is the average fraction of type τ bidders in n-bidder auctions. The second equality is obtained by exchanging the order of the summations.

For a given P_{n+1} and $P_n\subset P_{n+1}$, note that

$$\Pr(P_n|P_{n+1})=\frac{\#\{P_n^{P_{n+1}\backslash P_n}\}+1}{n+1},$$

$$\Pr(i=\tau|i\in P_n)=\frac{\#\{P_n^{\tau}\}}{n},$$

where P_n^τ represents the subset of P_n containing all bidders of type τ and $\#\{\cdot\}$ is the count operator. Thus, $\#\{P_n^{P_{n+1}\backslash P_n}\}$ denotes the number of times the bidder type randomly removed from P_{n+1} to obtain P_n appears in the set P_n.

The interior sum in the final expression in (10) can now be written as

$$\begin{aligned}
&\sum_{P_n\subset P_{n+1}}\Pr(i=\tau|i\in P_n)\Pr(P_n|P_{n+1})\\
&\quad=\frac{1}{n(n+1)}\sum_{P_n\subset P_{n+1}}\left(\frac{\#\{P_n^{P_{n+1}\backslash P_n}\}+1}{n+1}\right)\left(\frac{\#\{P_n^{\tau}\}}{n}\right).
\end{aligned}\tag{11}$$

Now fix a set P_{n+1} and let $\mathcal{A}$ denote the set of unique bidder types in P_{n+1}. Note that summing over all $P_n\subset P_{n+1}$ is equivalent to summing each of the types in $\mathcal{A}$. For any set P_n obtained by removing a type $a\neq\tau$ from P_{n+1}, $\#\{P_n^\tau\}=\#\{P_{n+1}^\tau\}$ and $\#\{P_n^{P_{n+1}\backslash P_n}\}=\#\{P_n^a\}=\#\{P_{n+1}^a\}-1$. Also, for the specific set P_n obtained by removing a type τ from P_{n+1}, $\#\{P_n^\tau\}=\#\{P_n^{P_{n+1}\backslash P_n}\}=\#\{P_{n+1}^\tau\}-1$. Therefore, (11) can be written

$$\begin{aligned}
&\sum_{P_n\subset P_{n+1}}\Pr(i=\tau|i\in P_n)\Pr(P_n|P_{n+1})\\
&\quad=\frac{1}{n(n+1)}\Big[\big(\#\{P_{n+1}^\tau\}-1+1\big)\big(\#\{P_{n+1}^\tau\}-1\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{a\in\mathcal{A},a\neq\tau}\left(\#\{P^a_{n+1}\}-1+1\right)\#\{P^\tau_{n+1}\}\Bigg]\\
&=\frac{\#\{P^\tau_{n+1}\}}{n(n+1)}\Bigg[\#\{P^\tau_{n+1}\}-1+\sum_{a\in\mathcal{A},a\neq\tau}\#\{P^a_{n+1}\}\Bigg]\\
&=\frac{\#\{P^\tau_{n+1}\}}{n(n+1)}\Bigg[-1+\sum_{a\in\mathcal{A}}\#\{P^a_{n+1}\}\Bigg]\\
&=\frac{\#\{P^\tau_{n+1}\}}{n+1}.
\end{aligned}
$$

Plugging the last expression into (10) yields the desired result:

$$
\begin{aligned}
&\sum_{P_n}\Pr(i=\tau|i\in P_n)\Pr(\mathcal{P}=P_n|N=n)\\
&\quad=\sum_{P_{n+1}}\Pr(i=\tau|i\in P_{n+1})\Pr(\mathcal{P}=P_{n+1}|N=n+1).
\end{aligned}
$$

□

In addition to providing intuition behind the meaning of Definition 4, Lemma 3 also provides a simple, testable implication of the condition that bidder types are independent of N. We apply this test below in our empirical application in Section 5. Note that rather than testing the implication of Definition 4 provided in Lemma 3, one could instead directly test the condition that bidder types are independent of N by examining all sets of participating bidders, but this test would be much more unwieldy.

Our next result shows that together these conditions imply $F_{n:n}=F^{n+1}_{n:n}$.

Lemma 4. *If bidder types are independent of N, then, for all n, $F_{n:n}=F^{n+1}_{n:n}$ if valuations are independent of supersets and $F_{n:n}\geq F^{n+1}_{n:n}$ if valuations are stochastically increasing in supersets.*

Proof. We prove the case where valuations are independent of supersets. The stochastically increasing case is analogous:

$$
\begin{aligned}
F_{n:n}(v)&=\sum_{P_n}\Pr(\mathcal{P}=P_n|N=n)F^{P_n}_{n:n}(v)\\
&=\sum_{P_n}\sum_{P_{n+1}\supset P_n}\Pr(P_n|P_{n+1})\Pr(\mathcal{P}=P_{n+1}|N=n+1)F^{P_n}_{n:n}(v)\\
&=\sum_{P_{n+1}}\sum_{P_n\subset P_{n+1}}\Pr(P_n|P_{n+1})\Pr(\mathcal{P}=P_{n+1}|N=n+1)F^{P_n}_{n:n}(v)\\
&=\sum_{P_{n+1}}\sum_{P_n\subset P_{n+1}}\Pr(P_n|P_{n+1})\Pr(\mathcal{P}=P_{n+1}|N=n+1)F^{P_n|P_{n+1}}_{n:n}(v)\\
&=\sum_{P_{n+1}}\Pr(\mathcal{P}=P_{n+1}|N=n+1)F^{P_{n+1}}_{n:n}(v)\\
&=F^{n+1}_{n:n}(v).
\end{aligned}
$$

The second equality follows because bidder types are independent of N, and the fourth equality follows because valuations are independent of supersets. □

Lemma 4 is the first result of which we are aware that demonstrates sufficient conditions for a setting to have valuations independent of N even when bidders are asymmetric. The assumption that valuations are independent of N is not used solely in AGQ but rather is central to identification in much of the empirical auctions literature (see, for example, Haile and Tamer (2003) or Sections 5.3 and 5.4 of Athey and Haile (2007)).[4] Lemma 4 demonstrates that this condition will be satisfied if valuations are independent of supersets and bidder types are independent of N.

3.4 *Bounds on buyer and seller surplus with bidder asymmetries*

We introduce some notation for our new upper bounds on buyer and seller surplus:

$$\underset{\sim}{F}_{n:n}(v) \equiv \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} E_{\mathcal{P}}\big(\phi_{\bar{n}}\big(F^{\mathcal{P}}_{\bar{n}-1:\bar{n}}(v)\big)^{\bar{n}} | N = \bar{n}\big), \tag{12}$$

$$\tilde{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v) - v_0 - \underset{\sim}{F}_{n:n}(r) \cdot (r - v_0), \tag{13}$$

$$\widetilde{BS}_n(r) \equiv \int_0^\infty \max\{r, v\}\, d\underset{\sim}{F}_{n:n}(v) - \int_0^\infty \max\{r, v\}\, dF_{n-1:n}(v). \tag{14}$$

Our next result combines Lemmas 1 and 4 above with results established in AGQ to yield the full set of bounds for private values auctions without assuming bidder exchangeability.

Theorem 1. *If bidders have private values, bidder types are independent of* N, *and Assumption* 3′ *holds, then, for any* $r \geq v_0$ *and for any* n *and* $\bar{n} > n$, *the following statements hold*:

(i) *If valuations are independent of supersets, then* $\pi_n(r) \in [\underline{\pi}_n(r), \tilde{\pi}_n(r)]$ *and* $BS_n(r) \in [\underline{BS}_n(r), \widetilde{BS}_n(r)]$. *Likewise,* $\max_r \pi_n(r) \in [\max_r \underline{\pi}_n(r), \max_r \tilde{\pi}_n(r)]$ *and* $\arg\max_r \pi_n(r) \in \{r : \tilde{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\}$.

(ii) *If valuations are stochastically increasing in supersets, then* $\pi_n(r) \leq \tilde{\pi}_n(r)$ *and* $BS_n(r) \leq \widetilde{BS}_n(r)$.

Proof. We prove (i); (ii) is analogous. From Lemma 4, $F^{n+1}_{n:n} = F_{n:n}$. AGQ's Lemma 4 implies that (2) holds. Lemmas 1 and 4 above imply that $F_{n:n}(v) \geq \underset{\sim}{F}_{n:n}(v)$, so $F_{n:n}(v) \in [\underset{\sim}{F}_{n:n}(v), \bar{F}_{n:n}(v)]$. Finally, we obtain the above bounds by applying AGQ's Lemma 2. □

[4] This type of independence has a variety of other names in the auctions literature, such as "exogenous participation" or "exogenous variation in the number of bidders" (Athey and Haile (2002, 2007)), or in some models corresponds to an absence of "selective entry" (Roberts and Sweeting (2016)).

Lemma 2 implies that our upper bounds on buyer and seller surplus are at least as small as AGQ's, as $\underset{\sim}{F}_{n:n}(v) \geq \underline{F}_{n:n}(v)$ for all v implies $\tilde{\pi}_n(r) \leq \bar{\pi}_n(r)$ and $\widetilde{BS}_n(r) \leq \overline{BS}_n(r)$. Also by Lemma 2, if for all v, $F^{\mathcal{P}}_{n-1:n}(v)$ is not almost surely constant with respect to $\mathcal{P}$, then for all v, $\underset{\sim}{F}_{n:n}(v) > \underline{F}_{n:n}(v)$. In this case our upper bounds are strictly smaller than AGQ's, that is, $\tilde{\pi}_n(r) < \bar{\pi}_n(r)$ and $\widetilde{BS}_n(r) < \overline{BS}_n(r)$. Interestingly, our bounds can be tighter even if F is exchangeable. Consider the case where bidders 1 and 2 have independent valuations distributed as $\exp(1)$, bidders 3 and 4 have independent valuations distributed as $\exp(2)$, and $\Pr(\mathcal{P} = \{1, 2\}) = \Pr(\mathcal{P} = \{3, 4\}) = 0.5$. Then F is a mixture of exchangeable distributions, and so is exchangeable. However for all v, $F^{\mathcal{P}}_{1:2}(v)$ depends on the realization of $\mathcal{P}$, and hence Lemma 2 implies our upper bounds will be tighter. The lower bounds on surplus are the same as in AGQ, as found in (5) and (7). For the bounds on the optimal reserve price, however, *both* the upper and lower bound will be tighter than those in AGQ, as both the upper and lower bound depend on the tighter upper bound on profits, $\tilde{\pi}_n$.

All the bounds—on buyer and seller surplus and the optimal reserve price—are sharp. This follows from the same logic as the sharpness of the AGQ bounds, as their symmetric setting is a special case of ours. Specifically, the upper bound on surplus will hold with equality in the symmetric IPV environment and the lower bound will hold with equality in a perfectly correlated private values environment.

We also find that exploiting bidder asymmetries is increasingly beneficial as the number of bidders increases. We state this as a corollary.

Corollary 1. *For any $r \geq v_0$ and any $\bar{n} > n$, $\bar{\pi}_n(r) - \tilde{\pi}_n(r)$ increases linearly in n for all $n \leq \bar{n}$.*

Proof. Applying the definition of $\bar{\pi}_n(r)$ and $\tilde{\pi}_n(r)$ from (6) and (13) yields

$$\bar{\pi}_n(r) - \tilde{\pi}_n(r) = \frac{n}{\bar{n}}(r - v_0)\left[E_{\mathcal{P}}\left(\phi_{\bar{n}}\left(F^{\mathcal{P}}_{\bar{n}-1:\bar{n}}(r)\right)^{\bar{n}} | N = \bar{n}\right) - \phi_{\bar{n}}\left(F_{\bar{n}-1:\bar{n}}(v)\right)^{\bar{n}}\right].$$

This expression only depends on n through the scaling factor at the beginning, yielding the desired result. □

Corollary 1 implies that the improvement of the upper bound on seller surplus, accounting for asymmetry over the original AGQ bound in (6) that ignores asymmetry, is increasing in n. This is important given that it is precisely when n is large, and thus close to $\bar{n}$, that the AGQ surplus bounds tend to be wide because there are fewer possible levels of the number of bidders between n and $\bar{n}$ to exploit when applying the order statistics relationship from (2).

As a final remark on these surplus bounds in Theorem 1, we note that one can also obtain bounds on surplus that do not rely on the assumptions of bidder types being independent of N and valuations being independent of (or stochastically increasing in) supersets. These alternative, wider bounds would simply use a single value of n, exploiting the bounds on $F_{n:n}$ derived in Lemma 1 in Section 3.2 rather than the tighter bounds derived in Section 3.3 that rely on variation in n.

4. Estimation

The method described herein for estimating bounds for the objects of interest follows closely that of AGQ, modified to allow for bidder asymmetries. We state the estimation approach for bounds on seller surplus; the bounds on buyer surplus are analogous.

We use a multiplicative kernel $K(\psi_1, \dots, \psi_5) = \prod_{c=1}^{5} k(\psi_c)$, where each $k(\cdot)$ is a quartic kernel given by $k(\psi) = b \cdot (s^2 - \psi^2)^2 \cdot \mathbb{1}\{|\psi| \leq s\}$. The support of $k(\cdot)$ is the compact set $[-s, s]$, and the constant b is calculated so that the kernel integrates to $\int_{-s}^{s} k(\psi)\, d\psi = 1$. We set $s = 25$ so that the density is always strictly positive, which implies that $b = 9.6 \times 10^{-8}$. We choose a bandwidth that eliminates the asymptotic bias of the estimator. Since the order of the quartic kernel we used is $r = 2$, and since we have $d = 5$ covariates, we set the bandwidth to be $h = T^{\frac{-1}{d+2r-1}}$, where T is the number of observations (for more details, see the final section of Chapter 11 of Hansen (2016)).

Letting $K(\xi/h) = K_h(\xi)$, we compute the cumulative distribution function (CDF) of the $(n-1)$th valuation at r given $X = x$ as

$$\hat{F}_{n-1:n}(r|x) = \frac{\displaystyle\sum_{t=1}^{T_n} K_h(X_t - x)\mathbb{1}\{B_t \leq r\}}{\displaystyle\sum_{t=1}^{T_n} K_h(X_t - x)}. \tag{15}$$

We use this expression to compute the upper bound from Lemma 1. To compute the lower bound, we average (15) across all partitions (combinations of bidder types) we observe for auctions with n bidders. Let Q_n be the total number of such partitions of size n. For partition q, let $part_t$ denote auction t's partition. We estimate the objects required for the lower bound as

$$\hat{F}^{q}_{n-1:n}(r|x) = \frac{\displaystyle\sum_{t=1}^{T_n} K_h(X_t - x)\mathbb{1}\{B_t \leq r\}\mathbb{1}\{part_t = q\}}{\displaystyle\sum_{t=1}^{T_n} K_h(X_t - x)\mathbb{1}\{part_t = q\}}, \tag{16}$$

$$\hat{E}_{\mathcal{P}}\big(\phi_n\big(\hat{F}^{\mathcal{P}}_{n-1:n}(r|x)\big)^n|N = n\big) = \sum_{q=1}^{Q_n}\left(\frac{\displaystyle\sum_{t=1}^{T_n}\mathbb{1}\{part_t = q\}}{T_n}\right)\hat{F}^{q}_{n-1:n}(r|x). \tag{17}$$

Thus, using (15) and (17) we obtain the bounds on the distribution of the maximum order statistic:

$$\begin{aligned}
\underset{\sim}{\hat{F}}_{n:n}(r|x) &= \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m}\hat{F}_{m-1:m}(r|x) + \frac{n}{\bar{n}}\hat{E}_{\mathcal{P}}\big(\phi_{\bar{n}}\big(\hat{F}^{\mathcal{P}}_{\bar{n}-1:\bar{n}}(r|x)\big)^{\bar{n}}|N = \bar{n}\big), \\
\hat{\bar{F}}_{n:n}(r|x) &= \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m}\hat{F}_{m-1:m}(r|x) + \frac{n}{\bar{n}}\hat{F}_{\bar{n}-1:\bar{n}}(r|x).
\end{aligned} \tag{18}$$

Finally, we obtain bounds on expected seller surplus as

$$\hat{\underline{\pi}}_n(r|x) = \hat{\Psi}_n(r|x) - v_0 - (r - v_0) \cdot \hat{\bar{F}}_{n:n}(r|x),$$
$$\hat{\bar{\pi}}_n(r|x) = \hat{\Psi}_n(r|x) - v_0 - (r - v_0) \cdot \hat{\underset{\sim}{F}}_{n:n}(r|x),$$

where v_0 is the seller's valuation for the good and

$$\hat{\Psi}_n(r|x) = \frac{\sum_{t=1}^{T_n} K_h(X_t - x)\max\{B_t, r\}}{\sum_{t=1}^{T_n} K_h(X_t - x)}.$$

For inference we adopt a simple, conservative approach. As in AGQ, we rely on pointwise inference throughout; developing a uniform inference approach for this estimation approach or the AGQ estimation approach is beyond the scope of this paper but would be an interesting avenue for future research. Rather than the asymptotic inference approach used by AGQ (which could be adapted to our estimator), we adopt conservative bands based on a nonparametric bootstrap, which we found to be computationally simpler to implement. At each estimated pointwise bound, we obtain the upper confidence band by taking the $1 - \alpha/2$ quantile of 200 nonparametric bootstrap replications of the upper bound and the lower confidence band given by the $\alpha/2$ quantile of 200 nonparametric bootstrap replications of the lower bound. These bands will be conservative by a simple Bonferroni-style argument.

5. Empirical application: U.S. timber auctions

To demonstrate the applicability of our approach we rely on data from U.S. timber auctions. Previous empirical auctions research has highlighted a natural asymmetry among bidders at these auctions: some bidders represent logging companies and others represent mills (Athey, Levin, and Seira (2011), Athey, Coey, and Levin (2013), Roberts and Sweeting (2016)). Mills have the capacity to process the timber whereas loggers do not, and mills typically have higher valuations than loggers. The bounds approaches of both AGQ and Haile and Tamer (2003) also focused on timber auction data and relied on the assumption that valuations are independent of N, but abstracted away from bidder asymmetries. Coey, Larsen, and Sweeney (2014) developed a test of the assumption that valuations are independent of N and, using a mixed sample of loggers and mills at timber auctions, rejected the assumption that valuations are independent of N. However, Coey, Larsen, and Sweeney (2014) demonstrated that within auctions where only loggers participated, the test fails to reject that valuations are independent of N, suggesting that accounting for bidder asymmetries is important in these settings so that required conditions hold for the validity of empirical auctions approaches, such as the bounds approaches of AGQ and Haile and Tamer (2003). We demonstrate below how exploiting these asymmetries can also improve and tighten bounds estimates when bidder identities or types are observed.

Table 1. Descriptive statistics for timber auction data.

Number of Bidders in Auction	Number of Observations	Average Fraction of Loggers
2	246	0.5915
3	250	0.616
4	244	0.6434
5	210	0.5971
6	137	0.635
7	102	0.647
8	59	0.6462

Our data are the same as those used in Coey, Larsen, and Sweeney (2014) and come from ascending auctions held in California between 1982 and 1989 in which there were at least two and less than nine entrants. We restrict the sample to auctions with winning bids lower than \$500 so as to exclude outliers, eliminating nine observations. The final sample consists of 1248 auctions. The data contain all bids, as well as auction-level information, such as appraisal variables, measures of local industry activity, and other sale characteristics. Table 1 displays, for each value of $n \in \{2, \dots, 8\}$, the number of auction observations in which n bidders participated as well as the average fraction of loggers in these auctions. Lemma 3 implies that one implication of the condition that bidder types are independent of N is that the average fraction of a given bidder type should be constant across different values of n. We test this with a simple F-test of the hypothesis of equivalence of the means in the final column in Table 1 and fail to reject that this is the case (p-value $= 0.254$).

We follow the approach described in Section 4 to estimate bounds on the distribution of the maximum order statistic and also bounds on the expected seller profit at various values of the reserve price. We set $\bar{n} = 8$. The controls we include are the estimated sales value of the timber (per unit of timber), estimated manufacturing and logging costs, the species concentration index (HHI), and the total volume of timber sold in the 6 months prior to each auction, first dividing each variable by its standard deviation. The units in the figures are dollars per thousand board feet.

Figure 1 displays the estimated bounds on the distribution of the maximum order statistic for different values of N ranging from 2 to 7 evaluated at the median values of the covariates (as are all results displayed in this paper). The figure displays the lower bound on this object using the approach described in AGQ, ignoring bidder asymmetries, as well as the approach proposed herein. Dashed lines represent pointwise, conservative 95% confidence bands as described in Section 4. As can be seen in each panel, the asymmetric approach provides tighter bounds at each value of n, illustrating two important implications of our identification results: first, ignoring bidder asymmetries, which may be necessary in some settings, for example, when information on bidder identities is unobserved, does not lead to inappropriately small bounds; instead, the bounds ignoring asymmetries are conservative. Second, when accounting for asymmetries is feasible, doing so will lead to strictly tighter bounds than ignoring these asymmetries. At

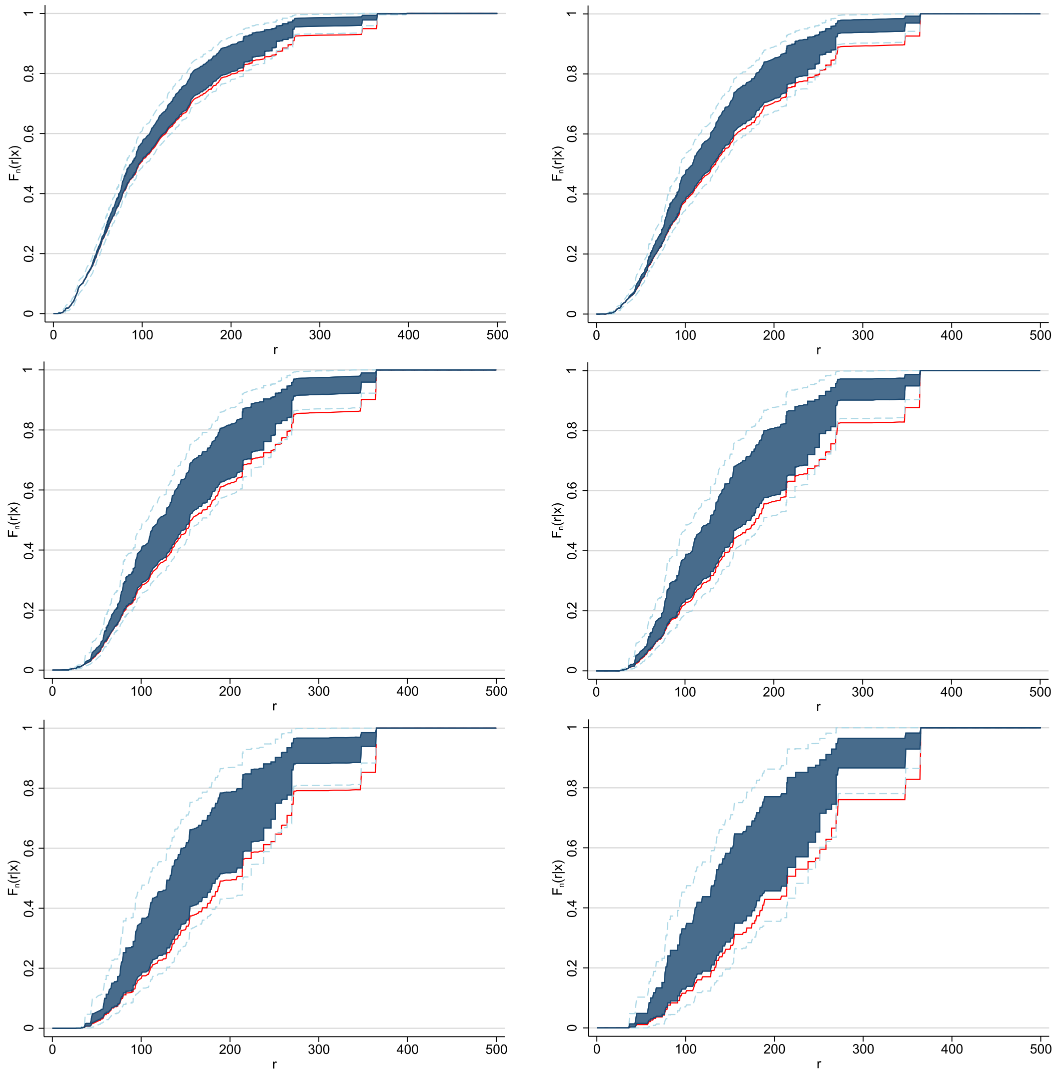

FIGURE 1. Bounds on the distribution of the maximum order statistic. *Notes*: Figure displays upper and lower bounds on the distribution of the maximum order statistic in n-bidder auctions for $n = 2, \dots, 7$. Filled-in area marks the region contained in the bounds that account for bidder asymmetries. Solid line represents lower bound on distribution that ignores bidder asymmetries. Dashed lines are bootstrapped pointwise 95% confidence bands.

some points in the support, the estimated lower bound ignoring asymmetry lies outside the confidence band surrounding the tighter estimates.

Figure 2 displays the estimated bounds on the expected profit function for different values of reserve price assuming that the seller's valuation, v_0, is 50.[5] Ignoring asymmetries among bidders yields strictly wider bounds on the expected profit function as it

[5]We choose this number because it is close to the actual reserve price reported in other timber auction settings (Haile and Tamer (2003)); the results do not differ qualitatively at different values of v_0.

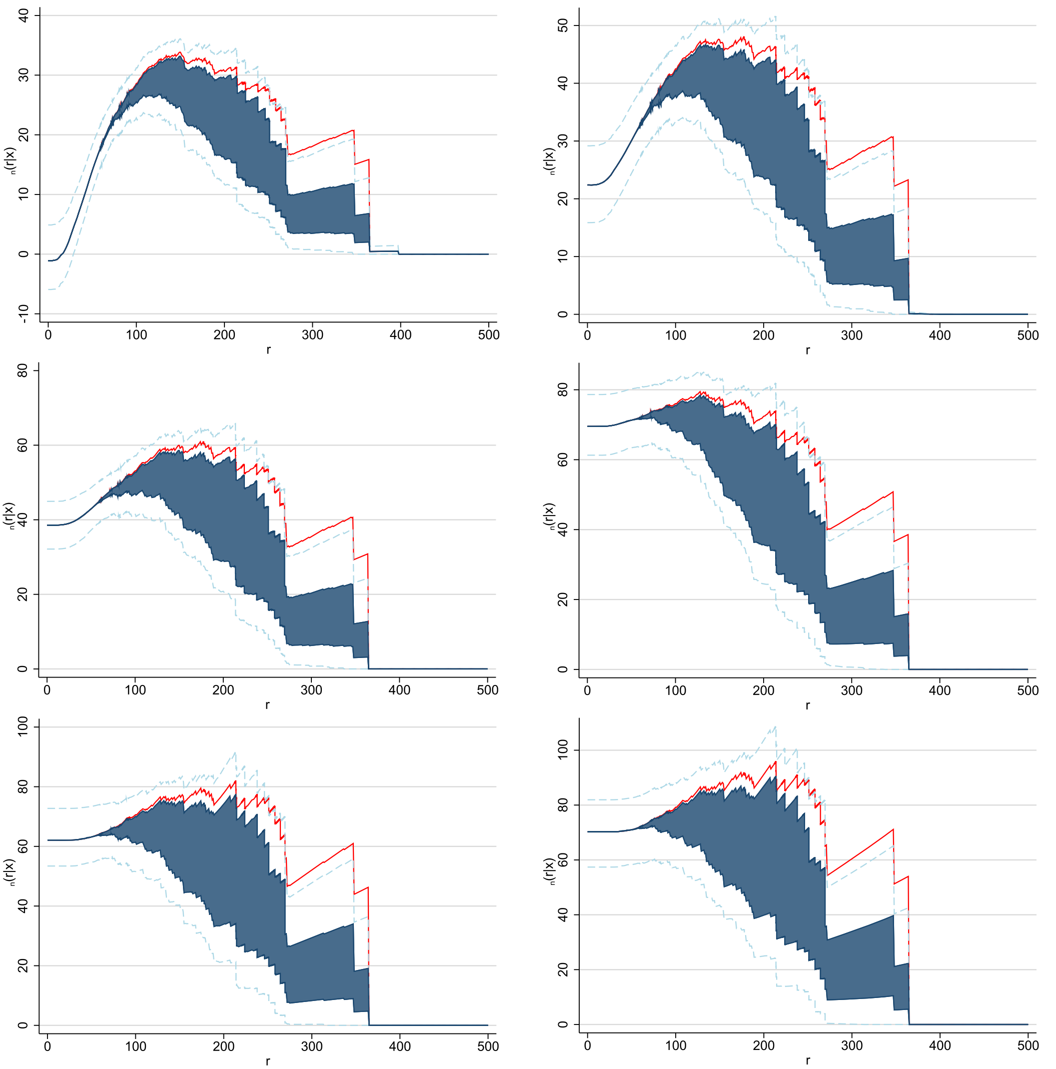

FIGURE 2. Bounds on the expected profit function. *Notes*: Figure displays upper and lower bounds on the seller surplus in n-bidder auctions for $n = 2, \dots, 7$ at different levels of the reserve price r (on the horizontal axis). Filled-in area marks the region contained in the bounds that account for bidder asymmetries. Solid line represents upper bound on surplus that ignores bidder asymmetries. Dashed lines are bootstrapped pointwise 95% confidence bands.

increases the upper bound on the profit function. The difference between the bounds ignoring asymmetries and those exploiting asymmetries appears to grow larger at larger values of r, where each of the bounds are also less precisely measured. As with the results in Figure 1, at some values of r, the estimated lower bound ignoring asymmetries lies beyond the 95% confidence band of the tighter bounds. This suggests that ignoring bidder asymmetries can lead to inaccurate conclusions as to which profits levels are achievable at a given reserve price.

TABLE 2. Bounds on optimal reserve price.

Number of Bidders in Auction	Optimal Reserve Bounds, Ignoring Asymmetries	Optimal Reserve Bounds, Exploiting Asymmetries
2	[90.5, 251.17]	[90.68, 223.87]
3	[81.76, 258.14]	[82.01, 237.95]
4	[72.15, 264.27]	[72.25, 237.95]
5	[60.62, 213.78]	[61.14, 177.89]
6	[59.73, 264.27]	[60.03, 246.21]
7	[62.02, 269.55]	[62.35, 246.21]

Note: The second column displays estimated lower and upper bounds on optimal reserve price when bidder asymmetries are ignored. The third column displays bounds that account for bidder asymmetries.

Table 2 displays bounds on the optimal reserve price which exploit bidder asymmetries and compare these bounds to those which ignore bidder asymmetries. Note that, unlike Figures 1 and 2, where exploiting bidder asymmetries only affected the lower or upper bound, respectively, but not both, in Table 2 *both* the upper and lower bounds on the optimal reserve are affected by the tightening that occurs due to averaging over different sets of participating bidders. For example, when $N = 2$, bounds on the optimal reserve price that would be suggested by ignoring bidder asymmetries are given by [91.36, 246.21]. Exploiting bidder asymmetries increases the estimated lower bound only slightly to 91.59 and decreases the upper bound dramatically to 214.60. Similar patterns are observed for other values of N.

We now turn to the question of how economically significant bidders' value correlation and asymmetry are. In the symmetric framework of AGQ, if the true information environment were one of independent private values, AGQ's upper bounds on buyer and seller surplus are attained: $\pi_n(r) = \bar{\pi}_n(r)$ and $BS_n(r) = \overline{BS}_n(r)$. In the more general asymmetric, correlated private value framework, $\pi_n(r) \leq \tilde{\pi}_n(r)$ and $BS_n(r) \leq \widetilde{BS}_n(r)$. It follows that $\bar{\pi}_n(r) - \tilde{\pi}_n(r)$ and $\overline{BS}_n(r) - \widetilde{BS}_n(r)$ are lower bounds on the amount by which allowing for asymmetries and correlation affects our estimates of buyer and seller surplus.[6] Figure 3 displays this lower bound for expected seller surplus in auctions with $n = 4$ bidders. The bound is monotonic over most of the range of reserves prices considered. As demonstrated in Corollary 1, this gap will be similar in shape for other values of $n \leq \bar{n}$, with n simply scaling the gap linearly.

Figure 3 implies that at a reserve price of \$200 per thousand board feet, which lies within the estimated bounds on the optimal reserve price from Table 2, correlation and asymmetries among bidders contribute to a loss in expected seller surplus of at least \$2.50 (approximately) per thousand board feet relative to what the seller would receive in a symmetric IPV setting. Given that Figure 2(c) demonstrates that the seller's surplus lies between about \$30 and \$60 at this reserve price, \$2.50 may represent an economically meaningful fraction of the seller's surplus. At a reserve price of \$250, on the other hand, this loss would be at least \$10 (approximately), representing a much larger fraction of

[6]By Lemma 2, these lower bounds must be nonnegative.

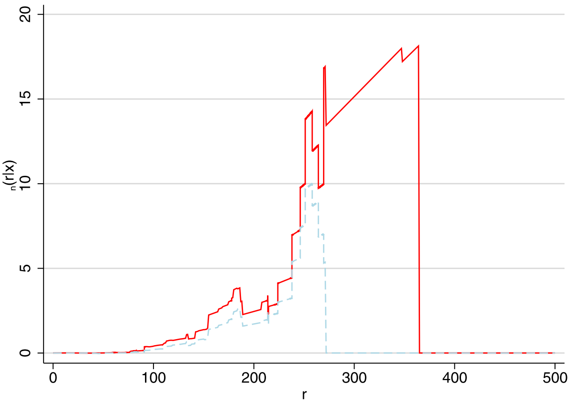

FIGURE 3. Lower bound on loss due to asymmetries/correlation, $\bar{\pi}_n(r|x) - \tilde{\pi}_n(r|x)$. *Notes*: Figure displays, in solid line, an estimate of $\Delta\pi_n(r|x) = \bar{\pi}_n(r|x) - \tilde{\pi}_n(r|x)$ in auctions with $n = 4$ bidders at different levels of the reserve price r (on the horizontal axis). Dashed line represents bootstrapped lower, one-sided pointwise 95% confidence band.

the surplus displayed in Figure 2. Interestingly, however, \$250 is a level of the reserve price that lies within the bounds obtained by ignoring asymmetries but is rejected by the approach exploiting asymmetries (see Table 2).

6. Conclusion

In private values ascending auctions when bidder identities or types are observable these asymmetries can provide useful information to improve estimated bounds on objects of interest. Thus, these asymmetries can help rather than hinder identification and estimation. When bidder identities or types are not observed but are still believed to differ, we provided sufficient conditions for bounds to remain valid. Some, but not all, of these results rely on a new condition we introduced that requires that the distribution of bidder types participating in the auction does not change as the number of bidders changes. We focused in this paper on the setting of Aradillas-López, Gandhi, and Quint (2013), but the ideas behind this approach can similarly be applied to other ascending auction settings, such as Haile and Tamer (2003), averaging over sets of participating bidders to obtain tighter bounds on distributions and surplus if bidder identities/types are observable, or demonstrating robustness of bounds to unobserved asymmetric types.

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